Permutation & Combination :-

Permutation :- Selection + Arrangement,

Combination :- Selection.

Fundamental principle of counting :-

1.) Multiplication principle :-

→ If an operation can be performed in m-diff. ways, following which a second operation can be performed in n-different ways, then the two operations in succession can be performed in $m \times n$ ways.

Q1 In a railway compartment, 8 seats are vacant on a bench. In how many ways can 5 passangers sit on them.

Sol^n

_ _ _ _ _ _ _ _ → 8 seats vacant,

No. of option for 1st passanger to occupy seat $= 8$

——— 2nd ——— $= 7$

——— 3rd ——— $= 6$

⋮

——— 5th ——— $= 8-4=4$

$\therefore \; 8 \times 7 \times 6 \times 5 \times 4 = 6720$ ways

2.) Addition Rule :-

⇒ If there are two jobs such that they can be performed independently in m & n ways respectively, then either of two jobs can be performed in (m+n) ways.

Q In a class 12 boys & 7 girls. In how many ways the teacher select either a boy or girl for competetion.

Sol^n The teacher has to perform, two jobs.

(i) Selecting a boy among 12 boys.

(ii) Selecting a girl among 7 girls.

→ Total no. of ways of selection $= 12 + 7 = 19$

Q How many no. lesser than 1000 can be formed from the digits 5,6,7,8,9, no reppatition.

Sol^n The no. of digits lesser than 1000 can be :

(i) No. of one digit, (–) 5 ways.

(ii) No. of two digit,

--→ $5 \times 4 = 20$ ways

(iii) No. of three digits,

---⇒ $5 \times 4 \times 3 = 60$ ways

∴ Total no. formed are :

$5 + 20 + 60 = 85$ no.s.

* Factorial Notation :-

The continued product of first n natural numbers (begining with 1 & ending with n) is called factoral of n.

It is denoted by $n!$ & $\lfloor n$

$$n! = n(n-1)(n-2) \cdots\cdots\cdots 3 \cdot 2 \cdot 1$$

$$n! = n(n-1)!$$

$$= n(n-1)(n-2)!$$

$$\vdots$$

→ Factorial of negative (-ve) integers and fractions are not defined.

→ It is only defined for whole numbers.

→ Factorial of zero is defined as equal to 1.

$$0! = 1$$

* $7! \rightarrow 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

$5! \rightarrow 5 \times 4 \times 3 \times 2 \times 1 = 120$

$4! \rightarrow 4 \times 3 \times 2 \times 1 = 24$

$3! \rightarrow 3 \times 2 \times 1 = 6$

$6! \rightarrow 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

$8! \rightarrow 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$

$9! \rightarrow 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362880$

$10! \rightarrow 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3628800$

Permutations,

→ In the arrangement of letters, groups or things, we are actually counting the different possible arrangements.

$$\left({}^nP_r = \frac{n!}{(n-r)!} \right) \forall (0 \le r \le n)$$

${}^nP_n = n!$

⇒ The no. of permutation of n distinct things taking r $(0 \le r \le n)$ at a time without repetition is defined by ${}^nP_r$ or by $P(n,r)$.

⇒ The number of permutation of n-different things taken r at a time, $0 \le r \le n$ & without repetition is

$n(n-1)(n-2) \cdots\cdots (n-r+1)$, which is denoted by ${}^nP_r$ or by $P(n,r)$.

⇒ The number of permutation of n distinct things taken r at a time, when repetition is allowed, is n^r.

⇒ The no. of permutation of n distinct things taken all at a time is $n!$.

⇒ In ${}^nP_r$, we count only those permutation in which repetition of things is not allowed.

⇒ The number of permutation of n things, of which k_1 are alike of one kind, k_2 are alike of 2^{nd} kind, ... k_r are alike of r^{th} kind such that $k_1 + k_2 + k_3 + \dots + k_r = n$, is

$$\frac{n!}{k_1! \cdot k_2! \cdots k_r!}$$

⇒ The no. of permutation of n things, p are alike of one kind, q are alike of 2nd kind & remaining all are distinct is

$$\frac{n!}{p! \cdot q!}$$

Q. In how many ways the letter of word GUNS can be arranged.

Sol^n Total no. of letter in GUNS $= 4$ all are diff.

$\therefore$ No. of arrangement $= {}^4P_4 = 4! = 24$

Q. How many no. lying b/w 1000 & 10000 can be formed from the digits 2, 3, 6, 7, 8, 9, without repetition.

Sol^n b/w 1000 & 10000 is a four digit number

_ _ _ _ 2, 3, 6, 7, 8, 9 $\Rightarrow$ 6 digits.

from 6 digits taken 4 at a time, ${}^6P_4$

$$\frac{6!}{2!} = 360$$

Q. How many permutation of the letters of the word BANANA are there?

Sol^n

$$\left.\begin{array}{l} B \to 1 \\ A \to 3 \\ N \to 2 \end{array}\right\} \text{Total} = 6 \qquad \therefore \text{No. of permutation} = \frac{6!}{3!\,2!} = 60$$

Q. Find the no. of arrangements of the letters of the word MADHUBANI. In how many of these arrangements,

(i) Do the words start with B.

(ii) Do all the vowels never occur together.

(iii) Do all the words begin with D & end in H.

Sol^n MADHUBANI → 9 letters,

A → 2 times,

∴ Total no. of arrangements $= \frac{9!}{2!} = 181440$

(i) Fix. B.

B(_ _ _ _ _ _ _ _) → 8 letters left,

∴ No. of arrangements $= \frac{8!}{2!} = 20160$

(ii) Vowels in MADHUBANI, are (AUAI) never occur together,

= Total no. of arrangement − no. of arrangment of vowel occur together

There are 9 places & vowel occur together 5 letter or consonant left & let that All letters of vowel is 1, then total no. of letters = 5 + 1 = 6

∴ No. of arrangement is. 6!

& no. of arrangement of vowels is $\frac{4!}{2!}$

∴ Total no. of arrangement of vowel occur together $= 6! \times \frac{4!}{2!} = 8640.$

⇒ Total no. of arrangement of letters vowels never occur together = 181440 − 8640 = 172800

(iii) The words begin with D & end in H, Let fix that,

D(_ _ _ _ _ _ _)H → 7 letter left

⇒ no. of arrangement $= \frac{7!}{2!} = 2520$

* Number of permutation under certain restriction:-

⇒ No. of permutation of n diff. objects taken r at a time, when a particular object always included in each arrangengent is $r \cdot {}^{n-1}P_{r-1}$.

⇒ No. of permutation of n diff. objects taken r at a time, when K particular object is never taken in each arrangement is

$${}^{n-K}P_r$$

⇒ No. of permutation of n- diff. objects taken all at a time, when m specified objects always occur come together is

$$m! \times (n-m+1)!$$

⇒ No. of permutation of n- different objects taken all at a time, when m- specified objects never come together is

$$n! - m! \times (n-m+1)!$$

⇒ No. of permutation of n-diff. objects, taken r at a time, when K particular object always included in each arrangement is,

$$K! \, [r-(K-1)] \, {}^{n-K}P_{r-K}$$

Circular Permutation $\left({}^{n}Q_r \text{ or } Q(n,r)\right)$

The circular permutation of n-diff. objects taken r at a time. is

$$ {}^{n}Q_r = \frac{n!}{r(n-r)!} = \frac{1}{r} \cdot {}^{n}P_r = {}^{n}C_r\,(r-1)! $$

$$ {}^{n}Q_n = \underline{(n-1)!} $$

→ Here clockwise & anticlockwise circular arrangement have be supposed different from one another.

→ In case of necklace or garland no distinction is made b/w clockwise & anticlockwise setting of beads, Hence total no. of permutation of n objects taken all at time is,

$$ \frac{(n-1)!}{2} $$

Rank of a word in a Dictionary :-

To find the Rank of a word in a dictionary arrange the words in alphabetical order.
In this dictionary the word may or may not be meaningful.

Q. Find the rank of RANK in dictionary?

Sol^n First arrange this in Alphabetical order,

AKNR

A   3!

K   3!

N   3!

RA   2!

RAK   1!

RANK   1!

Rank of RANK = 3!+3! 3! + 1!+1! = 20

Q. Find Rank of work INDIA.

Sol^n

ADIIN

A $\longrightarrow$ $\frac{4!}{2!} = 12$

D $\longrightarrow$ $\frac{4!}{2!} = 12$

IA $\longrightarrow$ 3! = 6

ID $\longrightarrow$ 3! = 6

II $\longrightarrow$ 3! = 6

INA $\longrightarrow$ 2! = 2

INDA $\longrightarrow$ 1! = 1

INDIA $\longrightarrow$ 1

Rank = 12+12 + 6+6+6 +2+1+1 = 46

Combinations :-

⇒ The no. of ways of selection of r objects out of n distinct objects is

$$\left({}^{n}C_{r} = \frac{n!}{r!(n-r)!} \right)$$

* ${}^{n}C_{r} = {}^{n}C_{n-r}$; $0 \leq r \leq n$

* If ${}^{n}C_{x} = {}^{n}C_{y}$, either $x=y$ or $x+y=n$

* ${}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$

⇒ Total no. of selection of n distinct objects is 2^n.

⇒ No. of ways of selection of r objects out of n identical objects is 1

⇒ The no. of ways of ~~ways~~ all possible selection of n identical objects is $(n+1)$.

⇒ The no. of ways of all possible selection of $n_1+n_2+n_3+n$ objects in which n_1 obj. are identical of Ist kind, n_2 are identical of 2nd kind, n_3 are identical of 3rd kind & n are distinct objects is

$$(n_1+1)(n_2+1)(n_3+1) \cdot 2^n$$

If at least an object is selected then total ways $= (n_1+1)(n_2+1)(n_3+1) \cdot 2^n - 1$

Special Uses of ${}^nC_r$.

→ No. of lines by n points in which no three points are collinear is ${}^nC_2$.

→ No. of lines by n points out of which m points are collinear & except these m points no three points are collinear, ${}^nC_2 - {}^mC_2 + 1$

→ No. of points of intersections of n lines if no three lines are concurrent and no two lines are parallel is ${}^nC_2$

→ No. of points of intersection of n lines out of which m lines are concurrent & except these m no any three are concurrent is ${}^nC_2 - {}^mC_2 + 1$

→ Maximum no. of point of ~~point of~~ intersection of n circles, $2.{}^nC_2$

→ Maximum no. of points of intersection of n lines & m circles, ${}^nC_2 + 2.{}^mC_2 + 2.{}^nC_1.{}^mC_1$

→ The no. of regions into which a set of n coplanar lines in general positive divide the plane is

$$\left(\frac{n(n+1)}{2} + 1\right)$$

→ Number of diagonal of a polygon of n sides

$$= {}^{n}C_{2} - n = \frac{n(n-3)}{2}$$

→ No. of triangle by n points in which no three are collinear $= {}^{n}C_{3}$.

→ No. of triangle form by n points out of which m points are collinear, ${}^{n}C_{3} - {}^{m}C_{3}$.

→ If a polygon of n sides & triangle are formed by joining the verteces then,

(i) No. of triangles having one side common with the side of polygon $n(n-4)$

(ii) No. of triangles having two side common with the side of polygon, n

(iii) No of triangles having no side common with the side of polygon ${}^{n}C_{3} - n - [n(n-4)]$

→ No. of parallelogram by n parallel lines of one set and m parallel lines of of another set is ${}^{n}C_{2} \cdot {}^{m}C_{2}$.

→ No. of rectangle of any size in a square of size $n \times n = {}^{n+1}C_{2} \times {}^{n+1}C_{2} = \sum_{r=1}^{n} r^{3}$.

→ No. of square of any size in $n \times n = \frac{n(n+1)(2n+1)}{6}$

→ No. of rectangle of any size in a rectangle of size $p \times n = {}^{p+1}C_{2} \times {}^{n+1}C_{2} = \frac{np(n+1)(p+1)}{4}$.

→ No. of square of any size in a rectangle of size $p \times n$ $(p < n)$ $= \sum_{r=1}^{p} (n+1-r)(p+1-r)$.

Divisors :-

1.) In general if $m = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots\cdots p_k^{\alpha_k}$ where p_i's are prime divisors of m, then total no. of all divisors of m,

$$m = (\alpha_1+1)(\alpha_2+1)(\alpha_3+1)\cdots\cdots(\alpha_k+1).$$

If 1 & m are excluded, then total no. of proper divisors,

$$= (\alpha_1+1)(\alpha_2+1)(\alpha_3+1)\cdots\cdots(\alpha_k+1) - 2$$

2.) The sum of all divisors of $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdots\cdots p_k^{\alpha_k}$ is equal to

$$\left(\frac{p_1^{\alpha_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{\alpha_2+1}-1}{p_2-1}\right)\left(\frac{p_3^{\alpha_3+1}-1}{p_3-1}\right)\cdots\cdots\left(\frac{p_k^{\alpha_k+1}-1}{p_k-1}\right)$$

and sum of proper divisors of m

$$= \left(\frac{p_1^{\alpha_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{\alpha_2+1}-1}{p_2-1}\right)\left(\frac{p_3^{\alpha_3+1}-1}{p_3-1}\right)\cdots\cdots\left(\frac{p_k^{\alpha_k+1}-1}{p_k-1}\right) - (m+1)$$

Q. Sum of all divisors of 720,

sol^n $720 \longrightarrow 2^4\, 3^2\, 5$

$$\left(\frac{2^5-1}{2-1}\right)\left(\frac{3^3-1}{3-1}\right)\left(\frac{5^2-1}{5-1}\right) = \frac{31}{1}\times\frac{26}{2}\times\frac{24}{4} = 2418$$

& the sum of all proper divisors of 720

$= 2418 - 721 = 1697$

3.) The no. of ways in which $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ ($p_1, p_2, p_3 \cdots p_k$ are primes) can be resolved as a product of two factors

$$= \begin{cases} \frac{1}{2}\left[(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_k)+1\right], & \text{if } n \text{ is perfect square} \\ \frac{1}{2}\left[(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_k)\right], & \text{if } n \text{ is not perfect square} \end{cases}$$

4.) The no. of ways in which a composite number $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ ($p_1, p_2, \cdots p_k$ are primes) can be resolved into two factors which are coprime to each other $= 2^{k-1}$, where k is the different prime factor of n.

Q→ Consider $N = 2^2 3^2 4^2 6^2 5^2$ & give the answers,

$N = 2^8 3^4 5^2$

(i) Total divisors,

$(8+1)(4+1)(2+1) = 135$

(ii) Total no. of divisor which divisible by 24,

$N = 2^8 3^4 5^2 = 24(2^5 3^3 5^2) = (5+1)(3+1)(2+1) = 72$

(iii) Find Total no. of divisors which are perfect square.

$(2^2)^4 (3^2)^2 (5^2)^1 = (4+1)(2+1)(1+1) = 30$

(iv) Total no. of divisors which are perfect cube.

$N = 2^8 3^4 5^2 = (2^3)^2 \cdot (3^3)^1 \cdot 2^2 3^1 5^2 = (2+1)(1+1) = 6$

Sum of the numbers formed by n-digits :

If repetition of digits is not allowed then

Sum of numbers = (sum of digits) $(n-1)!\left[\frac{10^n-1}{9}\right]$.

Q. How many 4 digit no. can be formed by 1, 2, 3, 4 without any repetition. Also find their sum.

Soln

Total No. of 4 digit no. formed = $4\times3\times2\times1 = 4! = 24$

$\Rightarrow (1+2+3+4)(1111)(^3P_3)$ ---> 1, 2, 3, 4

$= 10\times3!\times1111 = 60\times1111 = 66660$

Q. How many 10 digit no. can be formed using 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 without repetition and what is the sum of all such members?

Soln Total no. of 10 digit no. formed by 1, 2, 3, 4, 5, 6, 7, 8, 9, 0

$9\times9\times8\times7\times6\times5\times4\times3\times2\times1 = 9\cdot9! = 10!-9!$

Sum of the digits,

---------- fix

$(0+1+2+3+4+5+6+7+8+9)(1111111111)(^9P_9) -$

0(---------) fix.

$(1+2+3+4+5+6+7+8+9)(111111111)(^8P_8)$

Divisions into groups :-

The no. of ways to distribute n different objects b/w two groups which contain p & q objects respectively, when $p+q=n$ is,

$$^{n}C_{p} \cdot {}^{n-p}C_{q} = \frac{n!}{p!\,q!} \quad , \; p \neq q$$

In general,

The no. of ways to distribute n different objects b/w k groups which contain $n_1, n_2, \ldots\ldots n_k$ objects, where $n_1+n_2+n_3+\ldots\ldots+n_k = n$, is

$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots\ldots n_k!} \quad , \; n_i \neq n_j$$

(*) Equal divisons of objects :-

The no. of ways in which mn objects (diff.) are to be divided into m grops, each containing n objects and the order of group is not important is given by $\frac{(mn)!}{m!\,(n!)^m}$.

But ~~th~~ when the order of group is important then the no. of ways is given by

$$\frac{(mn)!}{(n!)^m}$$

Distribution into groups :-

1.) No. of ways in which n identical objects can be distributed into r different groups.

$$= \begin{cases} {}^{n+r-1}C_{r-1} & \text{if group may be empty} \\ {}^{n-1}C_{r-1} & \text{if no group is empty} \end{cases}$$

2.) The no. of ways in which n identical objects can be distributed into r groups so that no group contain less than p things & more than q things $(p<q)$ is

$$= \text{coefficient of } x^n \text{ in } \left(x^p + x^{p+1} + x^{p+2} + \ldots\ldots + x^q\right)^r$$

3.) No. of ways in which n different objects can be distributed into r different groups (when no group is empty) is

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n + {}^rC_3(r-3)^n + \ldots\ldots$$

$$\ldots\ldots(-1)^{r-1}.\ {}^rC_{r-1}$$

$\Rightarrow$ No. of combination of n-diff. object taken r at a time, when k particular object always occur, $\left({}^{n-k}C_{r-k}\right)$

$\Rightarrow$ No. of combination of n-diff. object taken r at a time, when k particular object never occur together,

$\left({}^{n-k}C_r\right)$

$\Rightarrow$ Total no. of combination of n different objects,

$\left(2^n - 1\right)$

Arrangement into groups :-

No. of ways in which n different objects can be arranged in r different groups is

$$= \begin{cases} {}^{n+r-1}P_{r-1} \;, & \text{group may have zero element.} \\ n!\, {}^{n-1}C_{r-1} \;, & \text{no group is empty} \end{cases}$$

Number of Integral solution of an equation :-

1.) No. of non-negative integral solⁿ of

$x_1 + x_2 + x_3 + \dots + x_r = n$ is ${}^{n+r-1}C_{r-1}$ or ${}^{n+r-1}C_n$

2.) No. of positive integral solution of

$x_1 + x_2 + x_3 + \dots + x_r = n$ is ${}^{n-1}C_{r-1}$ or ${}^{n-1}C_{n-r}$

3.) $xyz = N = x_1^{p_1} \cdot x_2^{p_2} \cdot x_3^{p_3}$ where x_1, x_2, x_3 are prime, then no. of positive integral solution of x, y, z is

$$= {}^{p_1+3-1}C_{3-1} \times {}^{p_2+3-1}C_{3-1} \times {}^{p_3+3-1}C_{3-1}$$

Application of Multinomial Expansion :-

(i) Solution of linear equations :-

The no. of non-negative integral solⁿ of $x_1 + x_2 + x_3 + \dots + x_K = n$ is equal to the no. of ways of distributing n identical objects among K groups which is equal to ${}^{n+K-1}C_{K-1}$.

The same can also be obtained by equating the coefficient of 'x^n' in expansion of

$$\cancel{x_1 + x_2} \left(1 + x + x^2 + x^3 + \dots\right)^K$$

$= $ coefficient of x^n in $(1-x)^{-K}$

$= $ coefficient of x^n in $\left(1 + {}^{K}C_1 x + {}^{K+1}C_2 x^2 + {}^{K+2}C_3 x^3 + \dots\dots + \dots + {}^{n+K-1}C_n x^n + \dots\right)$

$$= {}^{n+K-1}C_n = {}^{n+K-1}C_{K-1}$$

(ii) The no. of ways of chosing of k objects out of p objects of Ist kind, q object of 2nd kind & r objects of 3rd kind is equal to the coefficient of x^k in the expansion

$$(1+x+x^2+\ldots\ldots+x^p)(1+x+x^2+\ldots\ldots+x^q)(1+x+x^2+\ldots\ldots+x^r)$$

If one object of each kind is always to be included, then the no. of ways of chosing k objects out of these objects is equal to x^k coeff. in the expansion

$$(x+x^2+x^3+\ldots\ldots+x^p)(x+x^2+x^3+\ldots\ldots+x^q)(x+x^2+\ldots\ldots+x^r)$$

(iii) The no. of arrangements / permutations of k objects out of which p object of Ist kind, q object of 2nd kind & r objects of 3rd kind is equal to coefficient of x^k in the expansion of

$$k!\left[1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\cdots+\frac{x^p}{p!}\right]\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\cdots+\frac{x^q}{q!}\right)\left(1+\frac{x}{1!}+\cdots\cdots+\frac{x^r}{r!}\right)$$